Distinguishing Modified Gravity from Dark Energy

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The acceleration of the universe can be explained either through dark energy or through the modification of gravity on large scales. In this paper we investigate modified gravity models and compare their observable predictions with dark energy models. Modifications of general relativity are expected to be scale-independent on super-horizon scales and scale-dependent on sub-horizon scales. For scale-independent modifications, utilizing the conservation of the curvature scalar and a parameterized post-Newtonian formulation of cosmological perturbations, we derive results for large scale structure growth, weak gravitational lensing, and cosmic microwave background anisotropy. For scale-dependent modifications, inspired by recent $f(R)$ theories we introduce a parameterization for the gravitational coupling G and the post-Newtonian parameter γ. These parameterizations provide a convenient formalism for testing general relativity. However, we find that if dark energy is generalized to include both entropy and shear stress perturbations, and the dynamics of dark energy is unknown a priori, then modified gravity cannot in general be distinguished from dark energy using cosmological linear perturbations.

I. INTRODUCTION

Cosmic acceleration has been revealed by measurements of the redshift-distance relation $\chi(z)$ where χ is the comoving radial distance. The Hubble expansion rate follows from $H(z) = (d\chi/dz)^{-1}$ (in units where $c = 1$). This determination assumes only that the observable universe is adequately described by the Robertson-Walker metric, an assertion that is testable empirically [1, 2] and does not imply the validity of general relativity (GR).

The inference of dark energy follows once the Einstein field equations of general relativity are imposed on the Robertson-Walker metric yielding the Friedmann equations. These equations imply that a stress-energy-momentum component with negative pressure is needed to explain cosmic acceleration. This substance may be vacuum energy (i.e., a cosmological constant, giving rise to the ΛCDM model) or a scalar field [3]. The dark energy equation of state for uniform expansion, $p(\rho)$, can be determined from measurement of $\rho(z)$, which itself follows from $H(z)$ combined with the first Friedmann equation. Measuring $w(z) = p/\rho$ is the primary goal of dark energy experiments [4].

Another possibility is that general relativity requires modification on large distance scales and at late times in the universe. In this case cosmic acceleration would arise not from dark energy as a substance but rather from the dynamics of modified gravity. Modified gravity is not particularly attractive theoretically, but the observed cosmic acceleration is so surprising that all plausible explanations should be considered.

Observations of the cosmic expansion history cannot distinguish dark energy from modified gravity [5, 6]. Testing gravity requires exploring the evolution of spatial inhomogeneity (e.g. [7, 8] and references therein). Modified gravity theories must pass tests within the solar system and in relativistic binaries [9]. They are expected to show significant departures from general relativity only on cosmological distance scales. The combination of cosmic microwave background anisotropy, weak gravitational lensing, and the growth of clustering of dark matter and galaxies provides an opportunity to discriminate between dark energy and modified gravity.

Performing cosmological tests of modified gravity requires a set of predictions. There are two approaches to generating these predictions. In the first, a theory is specified by its Lagrangian (or other fundamental description) which provides the equations of motion for both homogeneous expansion and cosmological perturbations. A class of theories can be specified by giving a Lagrangian with free parameters, e.g. $f(R)$ theories where R is the Ricci scalar [6].

A second approach is inspired by the parameterized post-Newtonian framework for solar system tests [10]. Here one begins with the solution of the gravitational field equations (i.e., the metric) instead of the Lagrangian. Several authors have recently adopted this framework or a similar one [11, 12, 13, 14, 15]. The difficulty here is to find a good "Newtonian" description in cosmology to which one adds "Post-Newtonian" parameters. Metric perturbations in the scalar sector governing the growth of cosmic structure are characterized by two spatial scalar fields, Φ (the Newtonian potential) and Ψ (the spatial curvature potential). Even if we introduce the relationship $\Psi = \gamma\Phi$ with Eddington parameter γ [16], there remains one unknown function of space and time. In the solar system case, by contrast, $\Phi = -GM/r$ is known to provide an excellent approximation to planetary dynamics.

On solar system scales, and even within galaxies, one can use test-particle orbits to determine Φ, and light rays to determine $\Phi + \Psi$. This comparison yields impressive limits on $|\gamma - 1|$ in the solar system [9]. However, on a scale of several kpc, gravitational lensing combined with stellar dynamics in elliptical galaxies yields a current best result $|\gamma - 1| = 0.02 \pm 0.07$ [17]. At the scale of Gpc where dark energy appears to drive accelerated expansion, there are no longer any bound test-particle orbits to measure Φ, so a different approach, based on cosmological perturbation theory, is needed.

Previous work in the cosmological parameterized post-Newtonian framework has either assumed that some of the Einstein field equations remain valid with modified gravity [12] or has examined the dynamics of individual theories, e.g. [18]. Neither approach is ideal. One would prefer to sample all possible theories in a broad class, and for each theory to constrain the potentials by a consistency condition that does not assume general relativity or any particular modification thereof.

Such a consistency condition was found recently in Ref. [19] for the long-wavelength perturbations of a Robertson-Walker spacetime. This result was derived assuming that gravity is described by a classical four-dimensional metric theory having a well-defined infrared limit (i.e., the theory is well behaved for very long wavelength perturbations). For practical application, assumptions must also be made about the background spatial curvature and entropy perturbations. Assuming an inflationary or equivalent origin of perturbations, long-wavelength isentropic perturbations are imprinted in the spatial curvature on a flat background. In general relativity, these spatial curvature fluctuations, represented by the gauge-invariant ζ field of Bardeen et al. [20] or the $\mathcal{R}$ field of Lyth [21], are time-independent in the long-wavelength limit. Ref. [19] presented a derivation of the conserved curvature perturbation (calling it κ) making no assumption about the field equations except that they have a well-defined infrared limit. Physically this means that curvature perturbations are small and that all waves propagate causally. In what follows, the curvature perturbation introduced in Ref. [19] will be called ζ although its definition differs from that of Ref. [20]. For long wavelength perturbations on a flat background, $\mathcal{R} = \zeta$.

In the long-wavelength limit all cosmological perturbations factorize into functions of time multiplying the curvature perturbation or its spatial derivatives. This is true in GR and in modified gravity theories that are well-behaved in the infrared limit. One might naively expect this factorization to hold only on scales larger than the Hubble length. In general relativity, however, signals in the scalar sector propagate at the speed of sound, not the speed of light [22], leading to conservation of ζ on scales larger than the Jeans length.

To satisfy solar system tests, modified gravity theories for cosmic acceleration must introduce a length scale L_G below which general relativity is recovered. This length scale might be associated, for example, with the dynamics of new scalar degrees of freedom. The value of this length scale, compared with the size of the systems investigated, plays a crucial role in characterizing the behavior of modified gravity theories.

If L_G is much smaller than the length scales over which linear cosmological structure formation is measured (e.g., $L_G = 1$ Mpc), then the factorization of cosmological perturbations on scales larger than the Jeans length remains valid. We denote this case scale-independent modified gravity. These theories are like GR in that the curvature perturbation is conserved for the relevant length scales. This condition yields a great simplification of the dynamics, reducing cosmological perturbations to quadratures.

In GR, waves propagating in the scalar sector travel only at the speed of sound, so that scale-dependence of transfer functions arises only below the Jeans length. Modified gravity theories, however, typically have additional fields supporting waves that travel at the speed of light. In this case, L_G is the Hubble length and the factorization of cosmological perturbations no longer holds. Theories of this type are called scale-dependent modified gravity theories. Now two quantities, γ and the gravitational coupling G_Φ (the generalization of Newton's constant in the Poisson equation for Φ), are needed to characterize gravity, and both will vary with length scale as well as with time. Even such complicated models can still be approximated by parameterizations, as we will discuss below. Ref. [11] found a way of bridging super- and sub-horizon modifications to GR. Our parametrization, while not as general, is simpler because it only involves a few free parameters and no free functions.

Because we do not start with a Lagrangian, we cannot explain cosmic acceleration. We take the cosmic expansion history as given from observations. Rather than providing a complete theory of modified gravity, we provide a framework for observational tests of gravity in cosmology.

This paper is organized as follows. Section II describes the curvature perturbation and its use to build scale-independent modified gravity theories. Section III works out the growth of structure on sub-horizon scales, shows that the Poisson equation is modified, and derives results for cosmic microwave background anisotropy and weak gravitational lensing for scale-independent modified gravity theories. Section IV then examines the sub-horizon behavior of a currently popular class of theories known as $f(R)$ models and shows that they are scale-dependent. Section VI considers the alternative hypothesis that dark energy has a peculiar stress tensor while gravity is governed by GR. Finally, results are summarized and conclusions are presented in Section VII.

II. GRAVITY AT LONG WAVELENGTHS

Our starting point is the perturbed Robertson-Walker metric in conformal Newtonian gauge [23]:

$$ds^2 = a^2(t)[-(1+2\Phi)dt^2 + (1-2\Psi)\gamma_{ij}dx^i dx^i] \ , \quad (1)$$

where t is conformal time, $a(t) = 1/(1+z)$ is the expansion scale factor, and $\gamma_{ij}(\mathbf{x}, K)$ is the three-metric for a space of constant spatial curvature K, e.g. $\gamma_{ij}dx^i dx^j = d\chi^2 + r^2(\chi, K)d\Omega^2$ where $r(\chi, K)\sqrt{K} = \sin(\chi\sqrt{K})$ for $K > 0$ and is analytically continued for $K \leq 0$. Note that different conventions appear in the literature for the metric perturbations: $\Phi = \Psi_{\rm Hu} = \psi$ and $\Psi = -\Phi_{\rm Hu} = \phi$ where $(\Psi_{\rm Hu}, \Phi_{\rm Hu})$ are the potentials of Ref. [11] and

(ψ, ϕ) are the potentials of Ref. [24]. Linear perturbation theory is assumed to be valid throughout this paper.

The evolution of the scale factor can depend, in principle, on any quantities characterizing the geometry and composition of the Robertson-Walker background, for example the spatial curvature K and the entropy density (or equivalently, parameters characterizing the equation of state). We neglect entropy perturbations and consider only curvature perturbations on a flat ($K = 0$) background. Assuming that the unknown gravity theory has a well-defined infrared limit obeying causality, long wavelength curvature perturbations should evolve like separate Robertson-Walker universes. In this case it is possible to transform to a new set of coordinates, $t \to t - \alpha(t)$ and $\chi \to \chi(1+\zeta)$ where ζ is constant and $\dot{\alpha} = \Phi + \Psi - \zeta$, such that the new line element is eq. (1) with $\Phi = \Psi = 0$ and having spatial curvature $K(1+2\zeta)$. Thus, ζ is one-half the spatial curvature perturbation. Enforcing the coordinate transformation leads to the consistency condition [19]

$$\frac{1}{a^2}\frac{\partial}{\partial t}\left(\frac{a^2\Psi}{\mathcal{H}}\right) + \Phi - \Psi = \left[\frac{1}{a}\frac{\partial}{\partial t}\left(\frac{a}{\mathcal{H}}\right) + \frac{K}{\mathcal{H}^2} + O(k^2)\right]\zeta \ , \tag{2}$$

where $\mathcal{H} = \dot{a}/a = aH$ and k is the comoving wavenumber. Although Ref. [19] states that large-scale shear stress is neglected in this result, in fact eq. (2) is valid for $k \to 0$ in general relativity (and presumably in modified gravity theories) even if shear stress is present. The curvature term $K/\mathcal{H}^2$ has been computed assuming the Friedmann equation is valid; in modified gravity theories this term might be different but it must vanish when $K = 0$. Hereafter we assume $K = 0$ and drop the curvature term.

Eq. (2) may be regarded as a definition of the curvature perturbation ζ for arbitrary theories of gravity. For long wavelengths, ζ is independent of time. Sound waves in the matter sector or wave propagation in the modified gravity sector cause ζ to change with time on small scales. These changes are implied by the neglected terms proportional to $k^2\zeta$ in eq. (2). For now we ignore such terms, in effect assuming that both the Jeans length and L_G are smaller than the cosmological scales of interest.

During the radiation-dominated era the Jeans length is comparable to the Hubble length, and ζ (and Φ and Ψ) is damped for scales smaller than the Jeans length. We assume that during this early period of evolution general relativity is an excellent approximation so that the damping is well described by the transfer functions computed using standard codes [24, 25]. When modified gravity becomes important at low redshift, the Jeans length has dropped to a few Mpc or less. In practice, we modify CMBFAST only for $z < 30$ and then do so in such a way as to enforce eq. (2) with ζ corrected from its primeval value using the GR transfer function at $z = 30$.

Assuming a well-defined infrared limit, the time and space dependence of perturbations must factorize for wavelengths longer than the Jeans length or L_G, e.g.

$$\Phi(\mathbf{k}, t) = F(a)\zeta(\mathbf{k}) + O(k^2\zeta) \tag{3}$$

where $\mathbf{k}$ is the wavevector. Factorization implies that the ratio of the two gravitational potentials depends only on time as $k \to 0$. Therefore we may write, for any causal theory of gravity having a well-defined infrared limit,

$$\Psi(\mathbf{k}, t) = \gamma(a)\Phi(\mathbf{k}, t) + O(k^2\zeta) \ . \tag{4}$$

In modified gravity theories, $\gamma(a)$ is the only degree of freedom important for long-wavelength scalar perturbations. The conditions of causality and a well-defined infrared limit greatly restrict the dynamics of modified gravity theories.

We now make the key assumption that the terms proportional to $k^2\zeta$ in eqs. (2)–(4) can be neglected not only on super-horizon scales $k < \mathcal{H}^{-1}$ but also, as they can be in general relativity, on sub-horizon scales down to the Jeans length. This assumption defines a class of theories we call scale-independent modified gravity models.

Under these assumptions, modified gravity is completely specified on large scales by the scale-independent function $\gamma(a)$. At high redshift when dark energy is unimportant we require $\gamma \to 1$ in order to retain the success of general relativity in explaining the cosmic microwave background anisotropy [26]. Thus we adopt the following parameterization for scale-independent modified gravity:

$$\gamma(a) = 1 + \beta a^s \ , \tag{5}$$

where β and $s > 0$ are constants. Eqs. (2)–(5) now give

$$\gamma F(a) = a^{-2}\mathcal{H}\gamma^{(1/s)} \int_0^a a\gamma^{-(1/s)} \frac{d}{da}\left(\frac{a}{\mathcal{H}}\right) da \ . \tag{6}$$

Changing the lower limit of integration introduces a rapidly decaying solution which we ignore.

In general relativity with negligible shear stress, $\gamma = 1$. When a component with constant equation of state parameter $w > -\frac{1}{3}$ is dominant, $a \propto t^n$ with $n = 2/(1+3w)$ yielding $F = (3+3w)/(5+3w)$. Thus, for long wavelengths the potential drops from $\Phi = \frac{2}{3}\zeta$ during the radiation-dominated era to $\Phi = \frac{3}{5}\zeta$ during the matter-dominated era.

Of greater interest here is the evolution of the potentials during the matter-dominated era with modified gravity parameterized by (5). Figure 1 shows the results for $s = 1$ and $s = 3$ as well as the GR case $\beta = 0$. The background expansion history is chosen to match GR with $\Omega_m = 0.284$ and a cosmological constant with $\Omega_\Lambda = 0.716$. The choice $s = 3$ matches Caldwell et al. in the limit $\beta \ll 1$ [12]. However, as we will see below, our results differ from theirs because they assumed the validity of some components of the Einstein field equations, while we instead required consistent causal evolution on large scales. This difference will be discussed further below in Section VI.

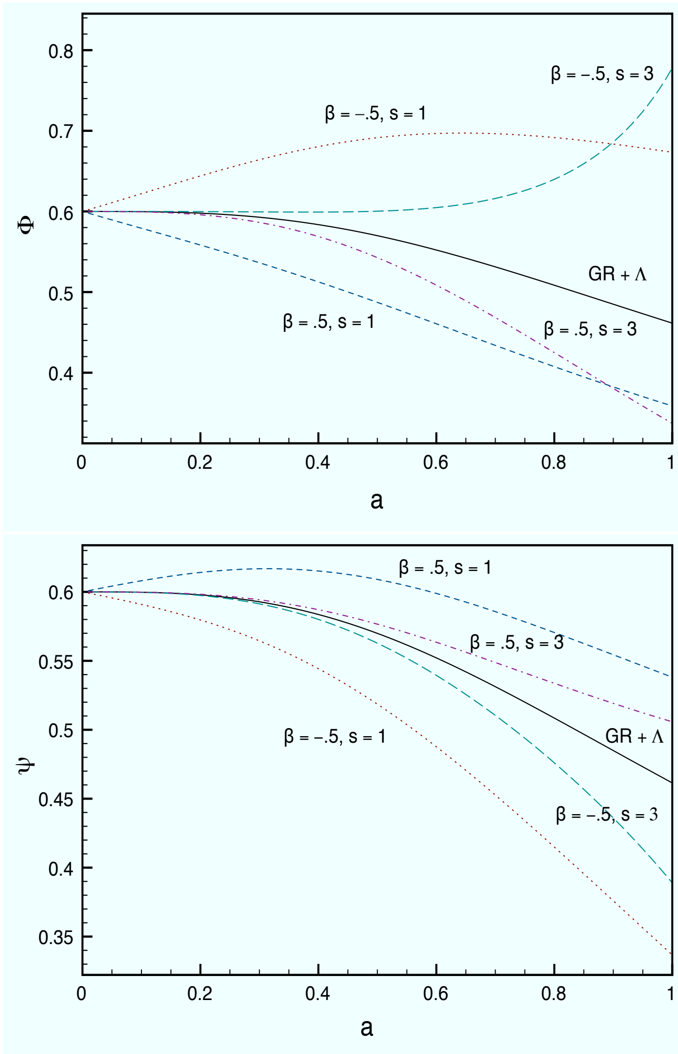

FIG. 1: Evolution of the scalar potentials Φ and Ψ in the long wavelength, scale-independent limit, assuming a $\zeta = 1$ normalization. Modified gravity effects, parameterized by eq. (5), arise later for larger s. The GR model assumes $\gamma = 1$ and a cosmological constant. For $\gamma < 1$ the Newtonian potential grows, unlike general relativity with a cosmological constant.

Figure 1 shows that the Newtonian potential Φ is enhanced and the spatial curvature $\Psi = \gamma\Phi$ is diminished for $\gamma < 1$, compared with general relativity. For $s = 3$ the modifications occur later because $|\gamma - 1|$ is smaller at earlier times. The quantitative results depend on the validity of eq. (2) but this qualitative behavior (the Newtonian potential being enhanced for $\gamma < 1$) should persist in general.

III. OBSERVABLES FOR SCALE-INDEPENDENT MODIFIED GRAVITY

With the time evolution of the metric in hand for long wavelengths we are now able to calculate observable quantities for scale-independent modified gravity theories parameterized by the constants (β, s). The effects considered here are the growth of structure in the dark matter, microwave background anisotropy, and weak gravitational lensing.

A. Growth of structure

Until now, no assumptions have been made about dynamics in the matter sector except for causality and consistency with a spatially homogeneous, uniformly expanding Robertson-Walker solution. To follow the growth of structure we must specify how the matter fields are coupled to gravity. Here we assume that the dark matter obeys the weak equivalence principle, i.e. collisionless dark matter particles follow geodesics. This choice explicitly forces scalar-tensor theories to the Jordan frame in which matter fields are minimally coupled to gravity.

In the conformal Newtonian gauge, on sub-horizon scales where $|\delta| \gg |\Psi|$ with $\delta \equiv \delta\rho/\rho_m$, and ρ_m is the average mass density, cold dark matter fluctuations obey the evolution equation

$$\ddot{\delta} + \mathcal{H}\dot{\delta} = -k^2\Phi \ . \tag{7}$$

This equation follows from particle number conservation and geodesic motion or, equivalently, from energy-momentum conservation. The density perturbation field can be written as

$$\delta(\mathbf{k}, t) = -k^2 D(a, k)\zeta_i(\mathbf{k}) \tag{8}$$

where ζ_i is the curvature perturbation at $a = a_i$ which we take to be $z = 30$ so that the Jeans length is smaller than the scales of interest and modified gravity has not yet become important. For $a > a_i$, Φ factorizes and eq. (7) can be reduced to quadratures for $D(a, k)$. With initial conditions $D(a, k) = D_i(k)$ and $\partial_t D(a, k) = \dot{D}_i(k)$ at $a = a_i$, the solution is

$$D(a, k) = D(a) + D_i(k) + a_i y(a)\dot{D}_i(k) \ , \tag{9}$$

where

$$D(a) \equiv y\int_{a_i}^{a} \frac{F}{\mathcal{H}}\, da - \int_{a_i}^{a} \frac{yF}{\mathcal{H}}\, da \ , \tag{10a}$$

$$y(a) \equiv \int_{a_i}^{a} \frac{da}{a^2\mathcal{H}} \ . \tag{10b}$$

The function $y(a)$ asymptotes to a constant but $D(a) \propto a$ for $a \gg a_i$ in the matter-dominated era. Thus the late-time solution for density perturbation growth factorizes,

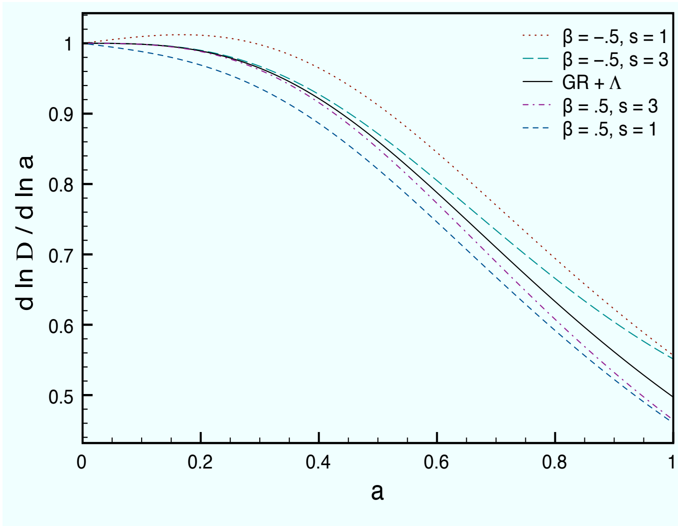

FIG. 2: Evolution of the logarithmic density perturbation growth rate $d\ln D/d\ln a$. Models with $\gamma < 1$ have enhanced growth relative to the ΛCDM (GR) model.

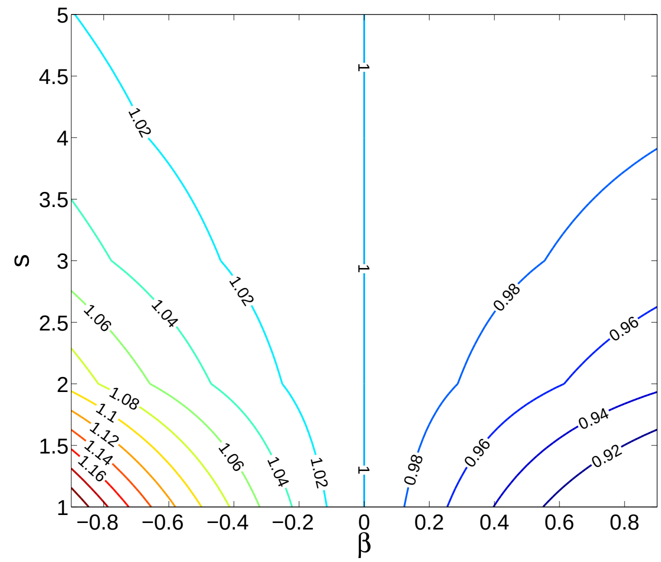

FIG. 3: Contour plot of $\sigma_8(\beta, s)/\sigma_8(\Lambda\text{CDM})$ with β running along the horizontal axis and s running along the vertical axis.

$D(a, k) = D(a)$ in eq. (8). This perturbation growth is often represented as a function of redshift by defining $g(z) \equiv D(a)/D(1)$ with $a = 1/(1 + z)$.

Figure 2 shows the logarithmic derivative of the density perturbation growth versus time for our parameterized modified gravity models. In the ΛCDM model, $d\ln D/d\ln a \approx [\Omega_m(a)]^{6/11}$ [27]. The transition to a cosmological constant-dominated universe leads to a suppression of growth. If instead gravity is modified, the growth rate can be increased or decreased relative to the GR case. The qualitative effects are easy to understand. We already saw that models with $\gamma < 1$ ($\beta < 0$) get an enhanced Newtonian potential Φ. A stronger potential increases the gravitational force on density perturbations leading to an enhanced growth rate.

The simplest test of growth of perturbations is the total perturbation growth by redshift zero, which is characterized by the variance of density fluctuations in spheres of radius $R_8 = 8\,h^{-1}$ Mpc,

$$\sigma_8^2 = \int_0^\infty \frac{d^3k}{(2\pi)^3} P_m(k) W^2(kR_8) \; , \tag{11}$$

where $W(x) = 3j_1(x)/x$. The power spectrum of matter density fluctuations is

$$P_m(k) = \frac{2\pi^2}{k^3}\Delta^2_{\mathcal{R}}\left(\frac{k}{k_0}\right)^{n_s-1} T_m^2(k, z = 30)\frac{D^2(z = 0)}{D^2(z = 30)} \; , \tag{12}$$

where $\Delta_{\mathcal{R}}$ is the amplitude of the initial scalar curvature fluctuations on scale k_0 and T_m is the transfer function for matter fluctuations in the synchronous gauge relative to $\zeta = \mathcal{R}$ computed by CMBFAST (which accounts for the suppression of growth during the radiation-dominated era). We adopt $\Delta^2_{\mathcal{R}} = 2.4 \times 10^{-9}$, $k_0 = 0.002$ Mpc^{-1}, and $n_s = 0.958$ [26].

Our modification of gravity is scale-invariant in that the k-dependence of the dark matter power spectrum is unchanged relative to GR. Thus the amplitude of density perturbations σ_8 depends on modified gravity only through the enhancement or diminution of growth shown in Fig. 2.

The specific results obtained here assume that the gravitational potentials factorize on scales larger than a few Mpc. If this is true, then CMB and galaxy clustering results on all scales can be fit by a single modified gravity theory with one function $\gamma(a)$ or, equivalently, $D(a)$. If, however, modified gravity introduces a length scale L_G intermediate between R_8 and the Hubble length, then the growth of structure will depend on wavenumber in a way not described by a scale-independent $D(a)$. In Section V below we consider an alternative scale-dependent parameterization of modified gravity and examine the observable consequences. For now, we assume $L_G < R_8$ or equivalently that super-horizon relations apply, as they do in GR, to sub-horizon scales all the way down to the larger of the Jeans and nonlinear lengths.

Figure 3 shows a contour plot of σ_8 (normalized to the GR case) for different choices of the modified gravity parameters. As expected, smaller values of γ (i.e., $\beta < 0$) lead to larger amplitude. Thus, modified gravity changes the amplitude of galaxy clustering relative to the CMB, and could explain any apparent discrepancy between the values of σ_8 inferred from CMB analysis and galaxy clustering or lensing measurements.

B. Modified Poisson Equation

Rearranging eqs. (3) and (8), we arrive at a modified Poisson equation relating the Newtonian potential to the overdensity:

$$\nabla^2\Phi = \frac{F(a)}{D(a)}\delta \equiv 4\pi G_\Phi(a) a^2 \rho_m \delta \ . \tag{13}$$

The space curvature potential obeys a similar Poisson equation,

$$\nabla^2\Psi = 4\pi G_\Psi(a) a^2 \rho_m \delta \ , \tag{14}$$

where $G_\Psi(a) = \gamma G_\Phi(a)$. Plots of G_Φ and G_Ψ are shown in Figure 4. Their time dependence is dominated by the potentials $F(a)$ and $\gamma(a)F(a)$ since δ is less sensitive to our modified gravity parameters (β, s). As a result, we see the same qualitative behavior as in Fig. 1. Models with $\gamma < 1$ produces a greater value of G_Φ relative to the ΛCDM model, and larger values of s produce larger late time behavior.

In GR, the gravitational coupling G is constant. In many alternative theories of gravity, the strength of gravity varies with time (and also with place, for length scales smaller than L_G). Time-varying G is well known for scalar-tensor theories [28, 29], but we find that it is a generic feature of all modified gravity theories with $\gamma \neq 1$ on cosmological scales.

The time-variation of G, represented by $\dot{G}/G$, has been severely constrained by measurements in the solar system, in stars, and in the early universe [30]. Limits on larger scales are provided by the microwave background [31]. The CMB acoustic peak structure will be unaffected if variations occur only long after recombination. Modified gravity explanations of cosmic acceleration suggest the need to constrain $\dot{G}/G$ on large length scales in the late universe. It would be very interesting to know, for example, to what extent the structure of clusters of galaxies and their X-ray emission can be used to constrain $\dot{G}/G$.

Note that the method used to derive our modified Poisson equations is roundabout. Had we started with a Lagrangian, the gravitational field equations would directly yield the gravitational coupling strength. Because we started with a phenomenological description of modified gravity, we instead deduce the dynamics of this coupling from the requirements of causal evolution and the weak equivalence principle. On small scales our treatment breaks down as modifications of general relativity must become scale-dependent. Nonetheless, the motivation to investigate limits on $\dot{G}/G$ on scales much larger than the solar system remains valid.

C. CMB temperature anisotropy

Modified gravity (or dark energy) affect the microwave background only at late times through the integrated

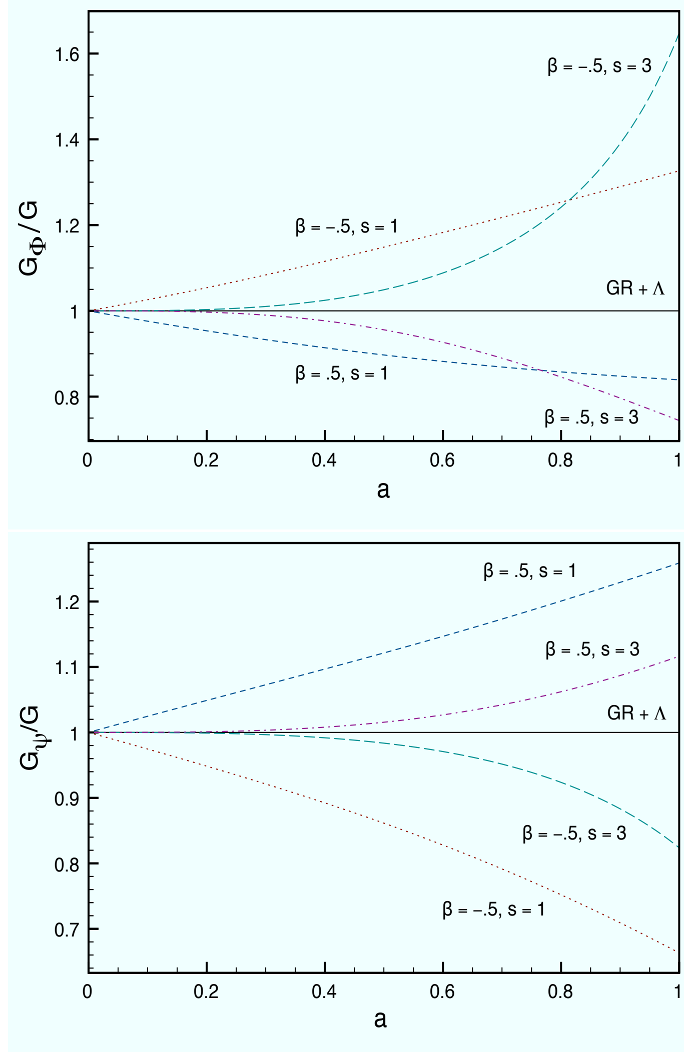

FIG. 4: Evolution of G_Φ and G_Ψ. The behavior is dominated by the potentials and exhibits the same features shown in Fig. 1.

Sachs-Wolfe (ISW) effect:

$$\frac{\Delta T}{T}(\hat{n}) = \int (\dot{\Psi} + \dot{\Phi})\, d\chi \tag{15}$$

where χ is the comoving radius and $\hat{n}$ is the photon direction. The ISW effect arises when the gravitational potentials change with time, as occurs during transitional periods in cosmic evolution. One such contribution occurs during the transition from radiation to matter domination. The other contribution is occurring today during the current transition to an accelerating expansion. The physics governing the matter-radiation transition is well explained by GR, while the physics governing the transition today is (for the models considered here) dependent on modified gravity parameters. Because the recent ISW effect arises relatively nearby ($z < 2$), it shows up only at large angular scales. The ISW contribution from some

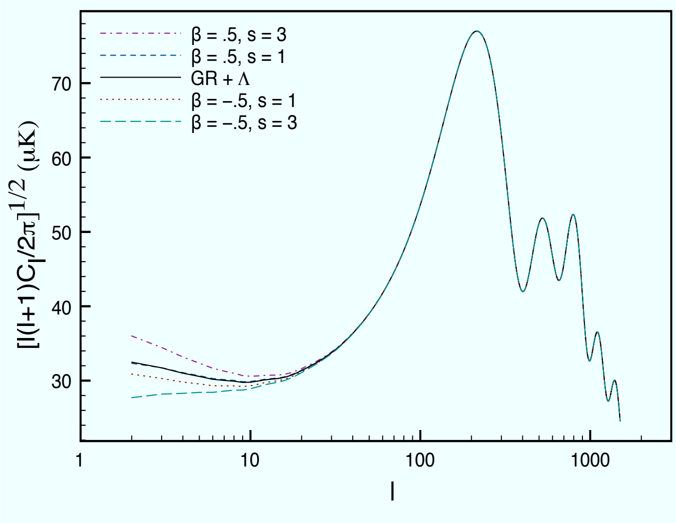

FIG. 5: The temperature anisotropy power spectrum for different parameter choices of modified gravity.

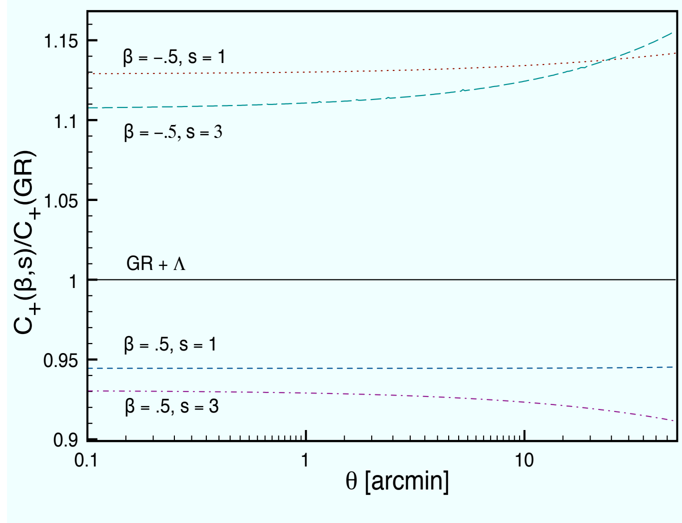

FIG. 6: Ratios of weak lensing shear correlation C_+, to its value for GR, for several sets of modified gravity parameters.

modified gravity models has been studied in several recent papers [6, 12].

We computed the CMB temperature anisotropy spectrum by modifying CMBFAST [25] to replace the ISW contribution of the ΛCDM model with that for our models assuming the factorization of the potential. The results are shown in Figure 5. As expected, only the low-order multipoles are affected. Models with higher s produce larger changes because they lead to larger time derivatives. Models with $0.2 < \gamma < 1$ $(-0.8 < \beta < 0)$ produce less anisotropy because of destructive interference between the ISW and primary anisotropy contributions. Although decreasing the quadrupole moment improves agreement with observations, little statistical weight can be given to this conclusion because the modifications, at least for $0.2 \leq \gamma \leq 1.5$, are smaller than cosmic variance. However, cross correlating the CMB with galaxy surveys could potentially be a more discriminating probe of modified gravity [32, 33]

It was recently found [34] that our results are consistent with recipe R1 of Caldwell et al. [12]. However, we expect differences from our Figure 5 for recipe R3 of [12] since ζ is not conserved on large scales in this scheme. This difference will be discussed below in Section VI.

D. Weak lensing

Metric perturbations $\Phi + \Psi$ affect both the energy of photons (ISW effect) and their direction of travel (gravitational lensing). Gravitational lensing causes both magnification (or de-magnification) and differential stretching (shear) of background images. The correlation function or power spectrum of weak gravitational lens shear is an observable measure of large-scale structure. The weak lensing power spectrum is given by [35]

$$P_l^\kappa = \int_0^{\chi_\infty} d\chi\, W^2(\chi) \frac{l^4}{\chi^4} P_{\Psi+\Phi}(k = \frac{l}{\chi}, \chi) \ , \tag{16}$$

where

$$\frac{P_{\Psi+\Phi}}{(1+\gamma)^2} = \frac{2\pi^2 \triangle_{\mathcal{R}}^2}{k^3} \left(\frac{k}{k_0}\right)^{n_s - 1} T_\Phi^2(k, z=30) \frac{F^2(z)}{F^2(z=30)} \ , \tag{17}$$

and

$$W(\chi) = \int_\chi^{\chi_\infty} d\chi' \frac{\chi' - \chi}{\chi'} \eta(\chi') \ . \tag{18}$$

Here χ_∞ is the comoving distance to $z = 10$ (the results change by a negligible amount if the maximum redshift lies anywhere between $6 \leq z \leq 15$), T_Φ is the transfer function of the Newtonian potential relative to ζ computed at $z = 30$ using CMBFAST and $\eta(\chi)$ is the radial distribution of sources, normalized with $\int \eta(\chi)\, d\chi = 1$. Note that these formulae assume a flat space. Our lensing analysis uses the source distribution

$$\eta(z) \propto z^2 \exp[-(1.41 z/z_{\rm med})^{1.5}] \tag{19}$$

with $z_{\rm med} = 1.26$ [36]. This distribution approximates the galaxy redshift distribution of the COSMOS survey, if there were no clumping.

Measurements of weak gravitational lens shear, for galaxies separated by angle θ on the sky, provide an estimate of shear correlation functions including

$$C_+(\theta) = \frac{1}{2\pi} \int_0^\infty P_l^\kappa J_0(l\theta) l\, dl \ . \tag{20}$$

Modifying gravity changes $F(z)$ thereby changing $C_+(\theta)$. Figure 6 plots $C_+(\theta)$ for modified gravity, normalized at

each θ by its GR value. At $z = 1$, 1 Mpc corresponds to 2.15 arcmin. Hence, for some of the scales shown in Figure 6 structures are nonlinear and thus beyond the regime of validity of the current framework. However, a scheme that takes a linear power spectrum to a nonlinear power spectrum would correct this flaw [37]. On such small scales, our assumption of scale-independent modified gravity may also be invalid, so Figure 6 should be regarded as suggestive, but not definitive, of modified gravity effects on weak lensing.

As expected, models with $\gamma < 1$ have larger shear correlations because they have a larger $F(z)$ and therefore more growth of structure (despite having a smaller $1+\gamma$). For angular scales less than about 10 arcmin, the effect is almost equivalent to a constant change in the normalization of the power spectrum, i.e., in the value of σ_8. At larger angular scales, the redshift-dependence of $F(z)$ at small redshift translates to a dependence on distance and hence on angular scale, however this is in a regime where the shear correlations are small and difficult to measure. Thus, scale-independent modified gravity theories predict an amplitude of weak lensing different from GR with the same CMB primary anisotropy. The acoustic peak amplitudes tightly constrain $\Delta_{\mathcal{R}}$. In principle, measurements of σ_8 based on the CMB acoustic peaks (which are unaffected by modified gravity) could differ both from measurements based on galaxy clustering [which depend on $D(z)$] and those based on weak lensing [which depend on $F(z)$]. Current error bars are inconclusive [26] but this comparison of different σ_8 values could eventually provide a powerful test of GR.

IV. COMPARISON WITH $f(R)$ THEORIES

Substantial work has already been done investigating modified gravity effects for $f(R)$ theories. Here we consider such theories where the Ricci scalar R is replaced by $R + f(R)$ in the Einstein-Hilbert action, and where the action is extremized with respect to the metric. In these models, the field equations are generically fourth-order. In effect, modified gravity introduces a new propagating scalar degree of freedom coupled to gravity, the scalaron $f_R \equiv df/dR$ [38]. The Compton wavelength of the scalaron imprints a physical length scale, which is made dimensionless by combining with the wavenumber k:

$$Q \equiv \frac{3k^2}{a^2}\frac{f_{RR}}{1+f_R} \ , \tag{21}$$

where $f_{RR} \equiv d^2f/dR^2$. Several papers have recently discussed cosmological perturbation evolution for metric $f(R)$ theories [39, 40, 41]; our notation most closely follows that of ref. [41] except our potentials Φ and Ψ are exchanged from theirs.

In $f(R)$ theories, ζ is conserved on super-horizon scales [11]. However, the scalaron obeys a nonlinear Klein-Gordon equation with two length scales: the Hubble length and the scalaron Compton wavelength. The interesting case for large-scale structure is the quasi-static regime of linear, sub-horizon perturbations ($k^2 \gg \mathcal{H}^2$) where [41]

$$\nabla^2\Phi \approx \frac{4\pi G a^2\rho_m}{1+f_R}\left(\frac{3+4Q}{3+3Q}\right)\delta \ , \quad \gamma \approx \frac{3+2Q}{3+4Q} \ . \tag{22}$$

Differentiating eq. (2) and substituting eqs. (7) and (22) along with the background evolution equations (5) and (6) of ref. [41] for a universe containing only nonrelativistic matter and (optionally) a cosmological constant yields

$$\dot{\zeta} = U\dot{\Psi} + V\Psi \ , \tag{23}$$

where

$$U \equiv \frac{a\mathcal{H}}{\Gamma^2}\frac{\partial}{\partial t}\left(\frac{\Gamma^2 B}{a\mathcal{H}^2}\right) + \frac{2QB}{3+2Q} \tag{24}$$

and

$$V \equiv \frac{4\pi G a^2\rho_m \Gamma B}{\gamma\mathcal{H}} + \frac{\partial}{\partial t}\left(\frac{B}{\gamma}\right) - \frac{\Gamma B}{\mathcal{H}a^2}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}\left(\frac{a}{\Gamma}\right)\right] \ , \tag{25}$$

where we have defined the auxiliary variables

$$\Gamma \ \equiv \ \frac{G_\Psi}{G} = \frac{1}{1+f_R}\left(\frac{3+2Q}{3+3Q}\right) \ , \tag{26a}$$

$$B \ \equiv \ a\left[\frac{\partial}{\partial t}\left(\frac{a}{\mathcal{H}}\right)\right]^{-1} = \frac{2(1+f_R)\mathcal{H}^2}{8\pi G a^2\rho_m + a^2\partial_t(\dot{f}_R/a^2)} \ . \tag{26b}$$

General relativity with a cosmological constant corresponds to the case $f = 2\Lambda$, $f_R = 0$, and $\gamma = \Gamma = 1$, yielding $U = V = 0$. Thus, ζ is conserved even on sub-horizon scales in a ΛCDM universe. However, this is no longer true if $f_R \neq 0$. Two distinct effects modify the curvature perturbation. First, the $1+f_R$ factor in (22) modifies the evolution on sub-horizon scales. In practice, this effect is small if $|f_R| \ll 1$, as is favored by galactic structure considerations [40]. In this case, the background expansion history is nearly identical to GR with a cosmological constant and gravity is significantly modified only at wavelengths approaching the scalaron Compton wavelength, where $Q \sim 1$. For long wavelengths such that $Q \ll 1$, $\gamma \approx 1 - \frac{2}{3}Q$ and the corrections introduced to eq. (2)–(4) by scalaron dynamics are $O(k^2)$. The treatment given in the preceding sections remains valid for $|f_R| \ll 1$ and $Q \ll 1$. However, this limit corresponds to general relativity. Unfortunately, the treatment presented in Section II, which was based on a scale-invariant modification of gravity, breaks down just where $f(R)$ theories begin to deviate significantly from GR.

The $f(R)$ models generically have $\gamma - 1 \propto k^2$ for sub-horizon wavelengths longer than the scalaron Compton wavelength. These models have a scale-dependent modification of gravity. Although we cannot use the results of

Section II to describe them, it is still possible to parameterize scale-dependent modified gravity models so as to obtain useful results for the sub-horizon growth of large-scale structure. A simple parameterization inspired by $f(R)$ theories is presented in the next section.

V. SCALE-DEPENDENT MODIFIED GRAVITY

For a wide class of theories, modified gravity leads generically to a Poisson equation with variable gravitational coupling. In the scale-invariant modifications of Section II, the Newton constant is replaced by the time-varying $G_\Phi(t)$ which follows from the scale-invariant potential ratio $\gamma(t)$. In scale-dependent modified gravity theories, on the other hand, $G_\Phi(k,t)$ and $\gamma(k,t) = G_\Psi/G_\Phi$ are functions of length scale as well as time, and there is no simple relation between them. Thus, more parameters are needed to characterize such theories [14].

Despite their generality, $f(R)$ theories with $f_R \ll 1$ have, for a wide range of sub-horizon length scales, a very simple form for G_Φ and γ given by eq. (22). To arrive at a simple phenomenological model we simplify the time dependence as follows:

$$\frac{G_\Phi}{G} = \frac{1+\alpha_1 k^2 a^s}{1+\alpha_2 k^2 a^s}\ , \quad \gamma = \frac{1+\beta_1 k^2 a^s}{1+\beta_2 k^2 a^s}\ . \tag{27}$$

We assume that these relationships hold only in the linear regime of cosmological density perturbations, and that $G_\Phi/G \to 1$ and $\gamma \to 1$ on solar system scales. We also require GR to hold at early times, implying $s > 0$. Eq. (27) describes $f(R)$ theories with $|f_R| \ll 1$ if $\alpha_1 = \frac{4}{3}\alpha_2 = 2\beta_1 = \beta_2 = 4f_{RR}/a^{2+s}$. As a simple post-Newtonian model we will now assume that $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are arbitrary constants with units of length squared. In order to ensure that G_Φ/G and γ are finite for all k, we require α_2 and β_2 to be non-negative. Moreover, we need $G_\Phi > 0$ in order to ensure that gravity is attractive. Hence, α_1 must be non-negative as well. This scale-dependent parameterization has a different dependence on length scale than that of Amin et al. [14]. It is chosen to reproduce the scale-dependence of $f(R)$ theories. For some modified gravity theories $\gamma = 1$, e.g., Einstein plus Yukawa gravity. For this model G_Φ/G in eq. (27) is multiplied by an overall factor α_2/α_1 [18] so that the deviation from Einstein gravity shows up only at large distances.

The class of theories considered here has at least three physical length scales: the horizon scale $a/\mathcal{H}$, the transition scale $a^{1+s/2}\sqrt{\alpha_1}$ where gravity changes strength (for simplicity, we consider models where the α_i and β_i are all of comparable magnitude), and the nonlinear length scale for structure formation (e.g., approximately 10 Mpc today). If $a^{1+s/2}\sqrt{\alpha_i}$ and $a^{1+s/2}\sqrt{\beta_i}$ are smaller than the nonlinear scale, then for purposes of large-scale structure formation, gravity is adequately described by GR. The parameterization of eq. (27) applies only to intermediate length scales between the horizon scale and the smaller transition scale $a^{1+s/2}\sqrt{\alpha_1}$. However, it implies that for long wavelengths and at early times, gravity reduces to GR (with constant gravitational coupling). This assumption can be relaxed at the cost of introducing additional parameters, which seems premature given the difficulty of measuring any post-Newtonian parameters. Also, for wavelengths short compared with $a^{1+s/2}\sqrt{\alpha_i}$ and $a^{1+s/2}\sqrt{\beta_i}$ but large compared with the nonlinear scale, the gravitational couplings are constant but differ from GR, e.g., $\gamma = \frac{1}{2}$ for $f(R)$.

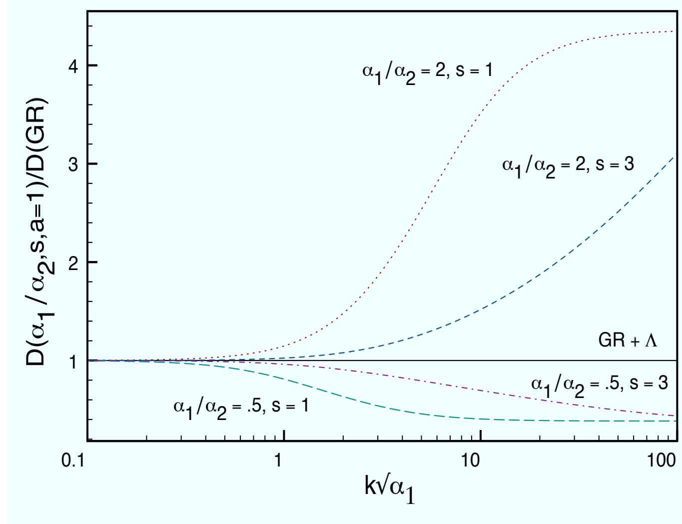

FIG. 7: The ratio of the density transfer function to the predicted GR case, plotted at $a = 1$ versus dimensionless wavenumber $k\sqrt{\alpha_1}$ for two values of α_1/α_2 and two values of s. For $a = 0.5$ the transfer functions are almost identical, except that $|D/D(GR) - 1|$ is reduced by approximately 13% for $\alpha_1/\alpha_2 = 0.5$ and 30% for $\alpha_1/\alpha_2 = 2$.

From the perspective of model testing, scale-dependent modified gravity is much more complicated than the scale-independent case considered in Sect. II. The models have four dimensional parameters plus an exponent giving the time dependence. However, the situation is not so bleak, because structure formation depends only on $G_\Phi(k,t)$ and not on $\gamma(k,t)$. In particular, matter density perturbations on scales larger than the Jeans length and smaller than the Hubble length follow from integration of

$$\ddot{\delta} + \mathcal{H}\dot{\delta} = 4\pi G_\Phi(k,t) a^2 \rho_m \delta\ . \tag{28}$$

At early times, $G_\Phi \to G$ and δ evolves as in the GR solution until the scale-dependent terms in eq. (27) become important. The density transfer function $D(k,t)$ given by eq. (8) is now scale-dependent at late times, implying a change in the shape of the matter power spectrum. It is easy to see that the transfer function can depend only on the dimensionless variables $(k\sqrt{\alpha_1}, \alpha_1/\alpha_2, a)$.

The most interesting new feature of scale-dependent modified gravity is the change of shape of the matter density transfer function. Figure 7 shows $D(k, a = 1)$

normalized to the GR result, obtained by numerically integrating eq. (28) with initial conditions given by the GR result $\mathcal{H}^2 D(a) \to \frac{2}{3}F = \frac{2}{5}$ for a matter-dominated universe at $a = 0.03$. As expected, at large length scales ($k\sqrt{\alpha_1} \ll 1$) the results converge to the GR limit. At short length scales, gravity is weaker than GR if $\alpha_1/\alpha_2 < 1$, leading to reduced growth; the growth is enhanced for $\alpha_1/\alpha_2 > 1$. Thus, scale-dependent gravity changes the shape of the matter power spectrum [42]. Ultimately, measuring this scale-dependence (and doing so at several redshifts) can constrain scale-dependent modified gravity theories. However, the interpretation of the galaxy power spectrum shape is complicated by scale-dependent biased galaxy formation and by the dark-matter-dependent transfer function (e.g., the neutrino fraction). Thus, while the linear growth of structure offers a potentially powerful test of GR versus modified gravity, it must be combined with other tests.

The second function characterizing scale-dependent modified gravity, $\gamma(k,t)$, is (in our analysis, which assumes no particular Lagrangian) unrelated to $G_\Phi(k,t)$. This function is best constrained by combining weak gravitational lensing and galaxy clustering measurements made at the same redshift. Care is required because the lensing amplitude is proportional to $(1+\gamma)\Phi$ while the galaxy density is proportional to D and also depends on biasing. Galaxy peculiar velocity measurements could be used, in principle, to reduce or ideally eliminate the dependence on biasing [7]. However, one must be careful not to assume the velocity-density relation obtained in GR. The continuity equation gives

$$\mathbf{v} = -\frac{i\mathbf{k}}{k^2}\frac{\partial \ln D}{\partial \ln a}\mathcal{H}\delta \ , \tag{29}$$

where the logarithmic growth rate $\partial \ln D/\partial \ln a$ is now scale-dependent, as shown in Figure 8. As in the case of galaxy clustering, measurement of this effect is contingent upon knowing the composition of dark matter (hot dark matter has a free-streaming scale, and its abundance determines the suppression of growth at small scales) and correcting for any velocity bias.

The greater freedom allowed by scale-dependent modified gravity models, and the fact that astrophysics (biased galaxy formation and dark matter dynamics) may also introduce scale-dependence into transfer functions, makes it challenging to test GR using growth of structure and weak gravitational lensing. It is likely that a combination of galaxy clustering, peculiar velocities, and weak lensing will be needed to obtain strong constraints on scale-dependent modified gravity theories.

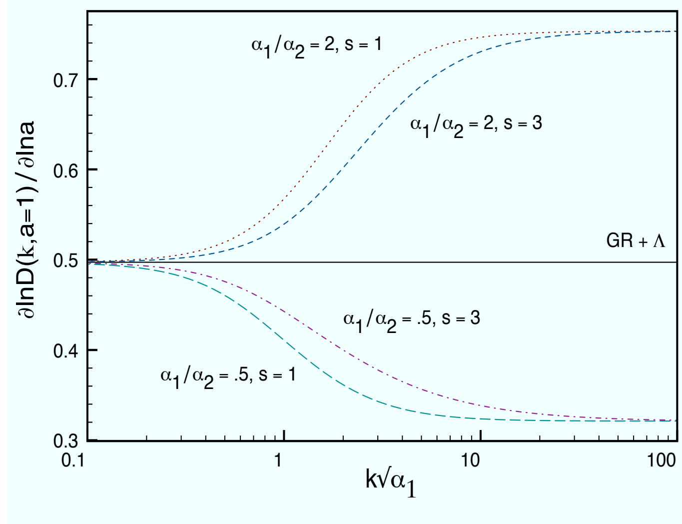

FIG. 8: Scale-dependence of the logarithmic density perturbation growth rate $\partial \ln D/\partial \ln a$ at $a = 1$ for scale-dependent modified gravity models.

VI. MODIFIED GRAVITY VERSUS SHEAR STRESS

A difference between the two longitudinal potentials Φ and Ψ need not signal modified gravity; it might arise from shear stress [13, 43]. For scalar mode fluctuations, the shear stress is fully characterized by a scalar potential π, such that the spatial stress tensor components are

$$T^i{}_j = p\delta^i{}_j + \frac{3}{2}(\bar\rho + \bar p)\left(\nabla^i\nabla_j - \frac{1}{3}\delta^i{}_j\Delta\right)\pi \tag{30}$$

where $(\bar\rho + \bar p)$ is the background enthalpy and $\Delta = \nabla^i\nabla_i$. In linearized GR, one of the Einstein field equations yields

$$\Psi - \Phi = 12\pi G a^2(\bar\rho + \bar p)\pi \ . \tag{31}$$

All of the results obtained in Sections II and III for modified gravity apply equally to GR with shear stress if γ is replaced by $12\pi G a^2(\bar\rho + \bar p)\pi/\Phi$.

In standard cosmology, the only significant source of shear stress is relativistic neutrinos after neutrino decoupling during the radiation-dominated era. For long wavelengths during the radiation-dominated era, neutrino shear stress gives [24]

$$\gamma - 1 = \frac{2}{5}\left(\frac{\rho_\nu + p_\nu}{\bar\rho + \bar p}\right) \ . \tag{32}$$

During the matter-dominated era, $\gamma - 1 \propto a^{-1}$ and shear stress is unimportant at late times in the ΛCDM model. It is also unimportant in simple quintessence models because shear stress vanishes for linear perturbations of a minimally-coupled scalar field.

Shear stress might nonetheless be important if cosmic acceleration is driven by an imperfect fluid. Without specifying the dynamics of this fluid, few constraints can be placed on π. One possible bound comes from the dominant energy condition, which states that each of the eigenvalues of $T^i{}_j$ must be smaller in absolute value than

the energy density. If this condition holds, then eqs. (30) and (31) can be combined to give rather weak bounds on $\Psi/\Phi - 1$.

Additional constraints follow from the initial-value constraints of GR and energy-momentum conservation, which for a spatially flat background become [19]

$$-k^2\Psi = 4\pi G a^2(\bar\rho+\bar p)(\delta + 3\mathcal{H}u)\ , \quad (33a)$$
$$\dot\Psi + \mathcal{H}\Phi = 4\pi G a^2(\bar\rho+\bar p)u\ , \quad (33b)$$
$$\dot\delta + 3\mathcal{H}\sigma = 3\dot\Psi - k^2 u\ , \quad (33c)$$
$$\dot u + (1-3c_s^2)\mathcal{H}u = c_s^2\delta + \sigma + \Phi - k^2\pi\ . \quad (33d)$$

The first of these equations is the usual Poisson equation in conformal Newtonian gauge. The density and velocity potential perturbations are defined from the energy-momentum tensor components by $T^0{}_0 = -\bar\rho - (\bar\rho+\bar p)\delta$, $T^0{}_i = -(\bar\rho+\bar p)\nabla_i u$, while the entropy perturbation is defined by $\sigma \equiv (\delta p - c_s^2\delta\rho)/(\bar\rho+\bar p)$ with $c_s^2 = d\bar p/d\bar\rho$. For a single perfect fluid like cold dark matter before its trajectories intersect, $\sigma = 0$. However, in general $\sigma \neq 0$ for a multi-component fluid, e.g. dark matter and a non-constant dark energy.

The freedom introduced by entropy and shear stress perturbations is, unfortunately, sufficient in principle to reproduce any observations of large-scale structure and gravitational lensing. Consider, for example, perfect measurements of galaxy peculiar velocities everywhere and at all times assuming that galaxies exactly trace cold dark matter. Then, eq. (33d) with $c_s^2 = \sigma = \pi = 0$ for CDM yields $\Phi(\mathbf{k},t)$. Assume furthermore that complete and ideal gravitational lensing measurements are available to yield $\Phi(\mathbf{k},t) + \Psi(\mathbf{k},t)$. Now, the GR initial-value constraints (33a)–(33b) suffice to yield $\delta(\mathbf{k},t)$ and $u(\mathbf{k},t)$ for the multi-component fluid of dark matter and dark energy. Requiring this fluid to obey energy-momentum conservation (33c)–(33d) yields σ and π. In short, perfect measurements of Φ and Ψ can be used to determine σ and π of the combined fluid of dark matter and dark energy. When combined with measurements of the dark matter density and velocity fields (assuming galaxies trace dark matter), one can, in principle, determine the energy-momentum tensor components for dark energy. One can think of this energy-momentum tensor for dark energy as the difference between the Einstein tensor for the observed metric and the energy-momentum tensor of the ordinary matter [11].

This approach can be used, in principle, to determine the dark energy entropy and shear stress needed to explain any observations of large-scale structure (including peculiar velocities) and weak lensing — even if there is no dark energy, but instead gravity is modified. In effect, the two observable metric fields Φ and Ψ can be exchanged for either σ and π (GR with exotic dark energy) or G_Φ and γ (modified gravity).

Although the initial-value constraints of GR can be used to determine the properties of dark energy, they cannot be used to model modified gravity. As we have seen in previous sections, modified gravity leads generically to a modified Poisson equation with a variable gravitational coupling $G_\Phi(k,t)$. In order not to break local Lorentz invariance by the selection of a preferred frame, the other components of the Einstein equation should also be modified. For example, in $f(R)$ theories the left-hand side of (33b) is multiplied by $(1 + f_R)$ while the right-hand side acquires an extra term $\frac{1}{2}\dot f_R$. Neglecting these modifications leads to violations of eq. (2) on large scales. For example, Caldwell et al. [12] modeled the dynamics of modified gravity with three recipes that assume the validity of some combination of eqs. (33a) and (33b). While recipes R1 and R2 satisfy the conservation of ζ, recipe R3 does not. As a result, it leads to a different prediction for the gravitational potentials and therefore the ISW effect.

VII. DISCUSSION

Parameterizing modified gravity theories in cosmology is much more difficult than parameterizing post-Newtonian gravity in the solar system because the gravitational potentials Φ and Ψ in the GR limit are not static Coulomb potentials in cosmology. In GR, on scales larger than the Jeans or nonlinear length scales, $\Psi = \Phi$ factorizes into a product of a time-dependent growth function and a spatially-varying curvature perturbation. If this factorization persists in modified gravity theories, then a simple post-Newtonian parameterization can be obtained. This is the approach we introduced in Section II. It has the practical virtue of yielding easily calculated predictions for all observables in the linear regime, including the growth of matter clustering, peculiar velocities, microwave background anisotropy, and weak gravitational lensing.

Introducing the Eddington parameterization $\Psi = \gamma\Phi$, where γ depends on time but not on space, we showed that structure grows faster (gravity is stronger) in models with $\gamma < 1$. However, the shape of the transfer functions (i.e., their dependence on spatial wavenumber) for these scale-independent models is unchanged compared with GR.

Unfortunately, realistic modified gravity theories such as the $f(R)$ models have scale-dependent effects and can no longer be described by only one post-Newtonian quantity γ. Instead, the strength of gravity, described by G_Φ in the Poisson equation, can vary independently of γ. Even so, because galaxy clustering and dynamics depends on G_Φ but not on γ, while weak lensing depends on γ, observations could, in principle, measure modified gravity parameters assuming there is no dark energy.

If dark energy is complex enough to require two additional fields to characterize its stress tensor (e.g., shear stress potential and entropy), then it appears that there is enough information in the dark energy model to account for any Φ and Ψ without modifying gravity. One cannot prove gravity is modified unless one can account for all significant contributions to the stress-energy tensor.

Thus, our hope to describe all modified gravity models with two parameters, yielding predictions measurably different from all dark energy models, has not been realized. Distinguishing modified gravity from dark energy will require making additional assumptions.

Nevertheless, the (β, s) parameterization of scale-independent modified gravity presented in Section II, and the $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ parameterization of scale-dependent modified gravity models presented in Section V, are still useful for characterizing observational data. If measurements of galaxy clustering, peculiar velocities, and weak lensing are all consistent with $\beta = 0$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, for example, then modified gravity and exotic dark energy models can both be excluded. If measurements require nonzero parameters, however, dark energy and modified gravity remain viable explanations until additional assumptions are made to distinguish them,e.g. restriction of the Lagrangian to a particular form.

A generic prediction of modified gravity theories in cosmology is that the gravitational coupling G_Φ in the Poisson equation should vary with time and with length scale. Departures from GR could be important not only in the linear regime of cosmological perturbations but perhaps also in the nonlinear regime (albeit on scales much larger than the solar system). Nonlinear effects may allow modified gravity to be distinguished from exotic dark energy, assuming that the dark energy fluctuations are small. For this reason it would be valuable to perform N-body simulations of structure formation using variable G_Φ, extending previous work [44] to the scale-independent and scale-dependent modified gravity models discussed in the current paper.

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